# COMPRESSIBLE-LIQUID MOTION IN POROUS MEDIUM WITH FLOWING AND STAGNANT REGIONS

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Filtration equations taking into account both flowing and stagnant regions of a porous medium [1] are solved for a semiinfinite region by the fractional-differentiation method [2, 3]. For a given arbitrary pressure change at the boundary of the region the filtration rate at this boundary may be determined.

## Formulation of the Problem

Earlier [1], phenomenological equations of liquid motion in a porous medium were proposed, taking into account the most significant structural feature of the filtrational flow – the presence of flowing and stagnant regions. Taking this structure into account must have a nontrivial effect on the pressure distribution in the liquid and the solid framework and on the magnitude of the liquid flux (in comparison with the generally adopted assumptions of [4], which do not take into account the filtrational-flow structure).

Consider the problem of describing the filtration of a compressible liquid in a semiinfinite region for a given change in the pressure $p_0(t)$ at the boundary [1]:

$$(1-\nu)\frac{\partial S_2}{\partial t}+\nu\frac{\partial S_1}{\partial t}=a\frac{\partial^2 S_1}{\partial x^2}, \tag{1}$$

$$\frac{\partial S_2}{\partial t}=\gamma(S_1-S_2),\ 0\leqslant x<\infty,\ 0<t<\infty, \tag{2}$$

$$\begin{gathered} S_1|_{x=0}=S_0(t),\ S_1|_{x=\infty}=0,\ S_2|_{x=\infty}=0,\\ S_1|_{t=0}=0,\ S_2|_{t=0}=0. \end{gathered} \tag{3}$$

Here $S_1 = P_1 - P_e$, $S_2 = P_2 - P_e$, where $P_1$ and $P_2$ are the liquid pressures in the flowing and stagnant regions; $P_e$ = const is the initial liquid pressure in the pore volume. Analogous equations describe the filtration in crackwise-porous media [4].

The pressure gradient at the boundary $(\partial P_1/\partial x)_{x=0} = (\partial S_1/\partial x)_{x=0}$ will be determined. This quantity determines, except for some constant factor, the liquidflow rate u at x = 0, since $u = -(k/\mu\varkappa)(\partial P_1/\partial x)$ [1].

## Determining Pressure Gradient at Boundary

Elimination of $S_2$ from Eq. (2), taking account of the initial condition, and substitution of the resulting equation into Eq. (1) gives the problem for determining $S_1$ in the form

$$\left\{\nu\frac{\partial}{\partial t}+(1-\nu)\gamma-(1-\nu)\gamma^2\int_0^t\exp[-\gamma(t-\tau)](\ldots)\,d\tau-a\frac{\partial^2}{\partial x^2}\right\}S_1(x,t)=0, \tag{4}$$

$$0\leqslant x<\infty,\ 0<t<\infty,$$

$$S_1|_{x=0}=S_0(t), \tag{5}$$

$$S_1|_{x=\infty}=0, \tag{6}$$

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$$S_1|_{t=0} = 0. \tag{7}$$

The system (4)-(7) is solved by an operational method. The expression for the pressure gradient at the boundary is

$$-\left.\frac{\partial S_1}{\partial \xi}\right|_{\xi=0} = \frac{1}{\sqrt{\pi}}\frac{d}{d\tau}\int_0^\tau \frac{1}{\sqrt{\tau-u}}\left\{\frac{d}{du}\int_0^u\left\{\exp\left[-\left(1+\frac{\beta}{2}\right)(u-z)\right]\times\right.\right.$$

$$\left.\left.\times I_0\left[\frac{\beta}{2}(u-z)\right]+(1+\beta)\int_0^{u-z}\exp\left[-\left(1+\frac{\beta}{2}\right)v\right]I_0\left(\frac{\beta}{2}v\right)dv\right\}S_0(z)\,dz\right\}du, \tag{8}$$

where $\xi = x(\nu\gamma/a)^{1/2}$; $\tau = \gamma t$; $\beta = (1-\nu)/\nu$.

Equation (8) is not convenient for analysis and numerical calculations, since it includes the operation of differentiation twice and an undetermined integral three times.

An expression is obtained for the gradient at the boundary by the method proposed earlier for the heat-conduction equation [2, 3, 5]. Consider derivatives of arbitrary order $\nu$:

$$D^\nu f(t) = \frac{d^\nu f(t)}{dt^\nu} = \frac{1}{\Gamma(1-\nu)}\int_0^t (t-\tau)^{-\nu} f(\tau)\,d\tau,\ \nu<1,$$

$$D^\nu D^\mu = D^{\nu+\mu},\ \nu+\mu\leqslant 1,$$

$$D^\nu f(t)\,g(t) = \sum_{n=0}^{\infty}\binom{\nu}{n}\frac{d^n f}{dt^n}\frac{d^{\nu-n}g}{dt^{\nu-n}},$$

$$D^\nu t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\nu)}\,t^{\mu-\nu}. \tag{9}$$

Taking account of Eq. (9), Eq. (4) may be rewritten in the form

$$\left\{e^{-\gamma t}\left[\frac{\partial}{\partial t}+\gamma(\beta-1)-\beta\gamma^2\frac{\partial^{-1}}{\partial t^{-1}}\right]e^{\gamma t}-\frac{a}{\nu}\frac{\partial^2}{\partial x^2}\right\}S_1 = 0. \tag{10}$$

Dimensionless variables will not be considered here in the main part of the work, since dimensionless variables do not offer the possibility of comparison with the well-known solutions for $\nu = 0$ and $\gamma = 0, \infty$.

By analogy with [2, 3, 5], Eq. (10) is written as the product of two operator factors, each of which contains only the first derivative with respect to the coordinate:

$$\left(L-\sqrt{\frac{a}{\nu}}\frac{\partial}{\partial x}\right)\left(L+\sqrt{\frac{a}{\nu}}\frac{\partial}{\partial x}\right)S_1 = 0. \tag{11}$$

It follows from a comparison of Eqs. (10) and (11) that the operator L is defined by the property

$$L^2 = e^{-\gamma t}\left[\frac{\partial}{\partial t}+\gamma(\beta-1)-\beta\gamma^2\frac{\partial^{-1}}{\partial t^{-1}}\right]e^{\gamma t}.$$

It may be established by direct verification that

$$L = e^{-\gamma t} M e^{\gamma t}, \tag{12}$$

and the operator M may be found by formal expansion of the square root

$$M = \sqrt{D+\gamma(\beta-1)-\beta\gamma^2 D^{-1}} = \sum_{k=0}^{\infty} a_k D^{\frac{1}{2}-k} \tag{13}$$

in powers of $D^{-1}$, so that $a_0 = 1$; $a_1 = \gamma(\beta-1)/2$; $a_2 = -(\beta\gamma^2 + a_1^2)/2$; $a_k = -(1/2)\sum_{n=1}^{k-1} a_{k-n}a_n$, $k\geqslant 3$.

Consider the equation formed by the right-hand factor in Eq. (11):

$$\left(L+\sqrt{\frac{a}{\nu}}\frac{\partial}{\partial x}\right)S_1 = 0. \tag{14}$$

As in [3], it may be shown that all the solutions of this equation are solutions of the initial Eq. (10) and, moreover, automatically satisfy Eqs. (6) and (7). Therefore, writing Eq. (14) with due regard to Eqs. (12) and (13) for x = 0 leads directly to the gradient at the boundary, in the form

$$-\sqrt{\frac{a}{\nu}}\left.\frac{\partial S_1}{\partial x}\right|_{x=0} = e^{-\nu t}\sqrt{D+\gamma(\beta-1)-\beta\gamma^2 D^{-1}}\,[e^{\nu t}S_0(t)]. \tag{15}$$

For $\nu = 1$ ($\beta = 0$), Eq. (15) gives

$$-\sqrt{a}\left.\frac{\partial S_1}{\partial x}\right|_{x=0} = e^{-\nu t}\sqrt{D-\gamma}\,e^{\nu t}S_0(t) = e^{-\nu t}(e^{\nu t}D^{1/2}e^{-\nu t})\,e^{\nu t}S_0(t) = D^{1/2}S_0(t),$$

which coincides with the well-known solution of the problem for elastic filtration conditions, disregarding the existence of flowing and stagnant regions [4].

For $\nu = 0$, Eq. (15) gives

$$-\sqrt{a}\left.\frac{\partial S_1}{\partial x}\right|_{x=0} = e^{-\nu t}\sqrt{\gamma-\gamma^2 D^{-1}}\,e^{\nu t}S_0(t) = e^{-\nu t}\sqrt{\gamma}\,D^{-1/2}\sqrt{D-\gamma}\,e^{\nu t}S_0(t) = \sqrt{\gamma}\,e^{-\nu t}D^{-1/2}e^{\nu t}D^{1/2}S_0(t),$$

which coincides with the result of [6].

For $\gamma = 0$, Eq. (15) gives a solution analogous to the case $\nu = 1$, but with an effective piezoconduction coefficient $a/\nu$.

## Example

Consider the usual case of a stepwise change in pressure at the boundary $S_1(-0,\ t) = S^* = \text{const}$. As in [6], it may be established that to the right of the boundary the pressure then changes according to the law $S_0(t) = S_1(+0,\ t) = [1-\exp(-\nu t)]S^*$. In fact, consider instead of Eqs. (1)-(2) the following equation for $S_1$:

$$\gamma\frac{\partial S_1}{\partial t}+\nu\frac{\partial^2 S_1}{\partial t^2}-a\gamma\frac{\partial^2 S_1}{\partial x^2}-a\frac{\partial^3 S_1}{\partial x^2\partial t}=0, \tag{16}$$
$$0\leqslant t<T,\quad -h\leqslant x<h,\quad h\to 0.$$

To obtain the condition at the boundary, the operator in Eq. (16) must be multiplied on the left by the function xt and integrated over the indicated region [6]. In the case when $\gamma \to \infty$, Eq. (16) transforms to the heat-conduction equation, and the condition at the boundary to the condition $S_0 = S^* = \text{const}$.

Let $\beta = 1$, i.e., $\nu = 1/2$. Then Eq. (15) had a particularly simple form. The coefficients of the operator M are then

$$a_{2n} = (-1)^n\binom{\frac{1}{2}}{n}\gamma^{2n},\quad a_{2n+1}=0.$$

Using Eqs. (14) and (9), and expanding in series of the exponential function, the pressure gradient at the boundary is found in the form

$$-\left.\frac{\partial S_1}{\partial \xi}\right|_{\xi=0} = \frac{S^* e^{-\tau}}{\sqrt{\pi}}\left(\frac{2}{1!!}\tau^{1/2}+\frac{0}{3!!}\tau^{3/2}+\frac{4}{5!!}\tau^{5/2}+\frac{8}{7!!}\tau^{7/2}+\frac{19}{9!!}\tau^{9/2}+\frac{16}{11!!}\tau^{11/2}+\frac{40}{13!!}\tau^{13/2}+\frac{80}{15!!}\tau^{15/2}+\right.$$
$$\left.+\frac{140}{17!!}\tau^{17/2}+\frac{224}{19!!}\tau^{19/2}+\frac{504}{21!!}\tau^{21/2}+\frac{1008}{23!!}\tau^{23/2}+\frac{1848}{25!!}\tau^{25/2}+\frac{3168}{27!!}\tau^{27/2}+\frac{6864}{29!!}\tau^{29/2}+\delta\right),$$
$$0<\delta<\frac{2^{15}\tau^{31/2}}{31!!}\left(1+\frac{\tau}{33}+\frac{\tau^2}{33\cdot 35}+\ldots\right)<\frac{2^{15}\tau^{31/2}}{31!!}\,\frac{1}{1-\tau/33}. \tag{17}$$

In contrast to Eq. (8), Eq. (17) is very convenient for computer calculations.

In the initial stages of the process $-\ \tau \ll (15/2)^{1/2}\ -$ the velocity of the boundary changes in proportion to $\gamma_a^{-1/2}t^{1/2}$ according to Eq. (17) and not in proportion to $a^{-1/2}t^{-1/2}$ (as in the case of [4], where stagnant regions were disregarded); this is due to the mass transfer between the flowing and stagnant regions.

## NOTATION

| | |
|---|---|
| $a$ | is the piezoconduction coefficient; |
| $a_n$ | are the numerical coefficients appearing in the solution; |
| $D^{\nu}$ | is the fractional-differentiation operator; |
| k | is the permeability of the medium; |
| L, M | are the operators; |
| P | is the pressure; |
| S | is the pressure difference; |
| u | is the velocity of liquid flow; |
| x, t | are the coordinate and time; |
| $\delta$ | is the absolute error; |
| $\gamma$ | is the mass-transfer constant between flowing and stagnant regions; |
| $\mu$ | is the liquid viscosity; |
| $\nu$ | is the fraction of flow-region volume in pore space; |
| $\varkappa$ | is the porosity referred to flowing-region volume; |
| $\xi$, $\tau$ | are the dimensionless coordinate and time; |
| $\beta$ | is the function of $\nu$. |

### Subscripts

| | |
|---|---|
| 1 | flowing regions; |
| 2 | stagnant regions; |
| 0 | boundary x = 0; |
| e | initial value. |

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